GRAPHS OF EXPLICIT FUNCTIONS.*

By Robert E. Bruce.

Graphs of Explicit Functions.

In elementary mathematics the explicit function plays a very important part. In spite of this fact the special methods of curve-tracing discussed in English text-books† are adapted more particularly to implicit functions. These methods may of course also be used with the explicit; but if one is interested mainly in the latter simpler methods are available.

As familiar illustrations of the use of the graphs of explicit functions in elementary work, two entirely dissimilar sets of problems may be mentioned the study of which is to some extent simplified by the use of methods described in this paper. The first of these sets is composed of the three famous problems of the Greeks. The cissoid, conchoid, and other curves used in the duplication of the cube, the trisection of the angle, and the squaring of the circle may be expressed as explicit functions and their graphs constructed without great difficulty. The second is the problem of solving for real roots rational integral equations of high degree and transcendental equations. The common graphic method of dealing with the transcendental equation, by breaking it up into two simultaneous equations, may of course also be applied to algebraic forms; and in either case the methods of construction here discussed may frequently be found useful in solving the problem. As an example the solution of

$$\text{arc}\sin x + \frac{1}{2^{x+1}} = 3$$

is given in Fig. 3. For reasons that are explained later the figure must be held upside down and the graph referred to O_{IV}

* Presented at the spring meeting, 1917, of the Association of Teachers of Mathematics in New England.

† For example, Frost or Johnson.

as origin. The equation has been broken up into the set

$$\begin{cases} y = \text{arc} \sin x, \\ y = 3 - \dfrac{1}{2^{x+1}}. \end{cases}$$

The x of the intersection is about $\frac{1}{2}$, which on substitution gives

$$\frac{5\pi}{6} + \frac{1}{\sqrt{8}} = 2.97.$$

It is not the intention of the writer to give the impression that the methods here discussed are new. They are, as a matter of fact, used to a limited extent in some of our textbooks; but so far as the writer is aware no adequate discussion of them is given. In addition let it be said that there is no thought of suggesting that these methods be substituted for those now in use. Their office is rather to supplement present methods; and when so used they will frequently enable the student to draw with comparative ease graphs that were practically beyond his power so long as he was limited to the elementary method of plotting points from a table of computed values of x and y.

In this elementary method the first step is algebraic. That is to say, values of y are built up from values of x by algebraic processes—the particular processes involved depending on the form of the function to be plotted. In the methods here discussed we substitute largely for this algebraic step the graphic step of building up the graph of the function in question from simpler graphs—the particular graphs used in any case depending upon the form of the function involved. It will be assumed then from the start that the student is familiar with certain simple graphs, just as he is familiar with the algebraic processes by means of which he finds the value of y from the value of x. The list given below contains all the forms that would ordinarily be used and is distinctly longer than the student needs at the start. Much can be accomplished with but a few forms.

List of Elementary Graphs.

1. $y = mx + b$.
2. $y = ax^2 + bx + c$.

3. $y = x^n$ for $n = 1, 2, 3, -1, -2$.
4. $y = n^x$ for $n = 2, e, 3$.
5. $y = \log_n x$ for $n = 10$ and e.
6. The trigonometric functions.
7. The arc functions.

As already indicated it is the purpose of the paper to discuss graphic methods for deriving from these simple forms the graphs of others that are more complicated. With this purpose in view we attack the following question the answer to which, as will be seen, supplies the methods demanded:

Given the graphs of $y = f(x)$ and $y = \phi(x)$ what relationships do the graphs of the following bear to them?*

1. $y = f(x) \pm \varphi(x)$:
$$y = f(x) \pm c,$$
$$y = f(x \pm c).$$
2. $y = f(x) \cdot \varphi(x)$:
$$y = cf(x), \quad y = -f(x),$$
$$y = f(cx), \quad y = f(-x),$$
3. $y = \dfrac{f(x)}{\varphi(x)} = f(x) \cdot \dfrac{1}{\varphi(x)}$:
$$y = \frac{1}{f(x)},$$
$$y = f\left(\frac{1}{x}\right).$$
4. $y = \sqrt{f(x)}$.
5. $x = f(y)$.

This list, it will be observed, is by no means exhaustive. In particular, $y = \sqrt{f(x)}$ is but one case under the more general form $y = [f(x)]^n$. The list is, however, sufficiently exhaustive for all ordinary purposes. No attempt will be made to study

* It will be observed that the use of the function notation, while convenient, is not absolutely necessary.

all the cases in detail. It is believed that no difficulty will be experienced in supplying the reasoning where it is omitted. As one would expect, the effective use of the methods depends to a large extent on the use of co-ordinate paper. Unfortunately it has not proved practicable to make this appear in the figures.

I. Given the graphs of $y=f(x)$ and $y=\phi(x)$ to find the graph of $y=f(x)+\phi(x)\equiv F(x)$. (See Fig. 1.)

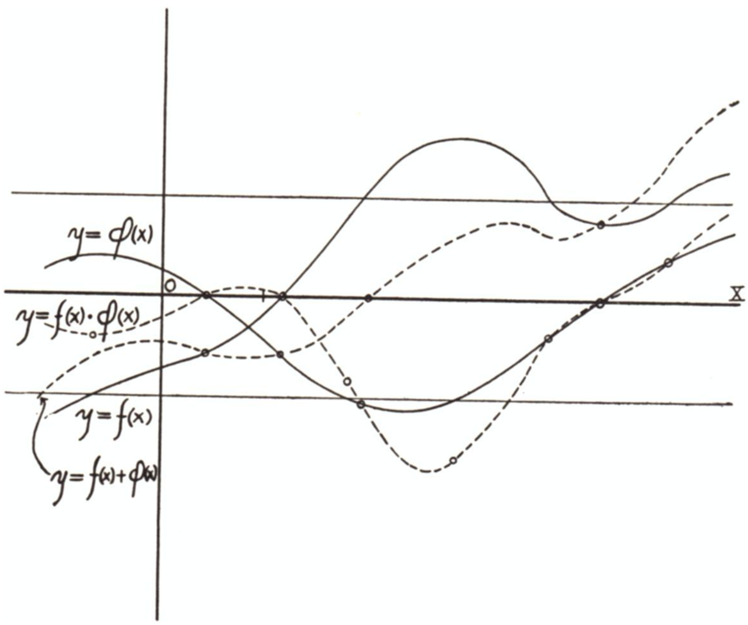

FIG 1.

Since the ordinates of $y=F(x)$ are the sums of the corresponding ordinates of the two given curves, it is clear that when either of the given curves crosses the x-axis, $y=F(x)$ crosses the other. $y=F(x)$ crosses the x-axis when the given curves are equally distant from the axis but on opposite sides of it. When the given graphs are on the same side of the x-axis, the graph of $y=F(x)$ is on the same side but at a greater distance from the axis. When the given curves are on opposite sides of the axis, the graph of $y=F(x)$ is between them. In addition to the points located at once from the considerations given above, as many more as desired may be located by adding

ordinates of one of the given curves to the corresponding ordinates of the other with the dividers.

II. Given the graphs of $y = f(x)$ and $y = \phi(x)$ to find the graph of $y = f(x) - \phi(x) \equiv F(x)$. (See Fig. 2.)

In this case the particular illustration $y = 2^x - x^3$ is given. Obviously, since the ordinates of $y = F(x)$ are the algebraic differences of the corresponding ordinates of the given curves, when the curves intersect $y = F(x)$ crosses the x-axis. The effects of the change of sign of x^3 and of the relative distances of the two curves from the x-axis are readily seen from the figure.

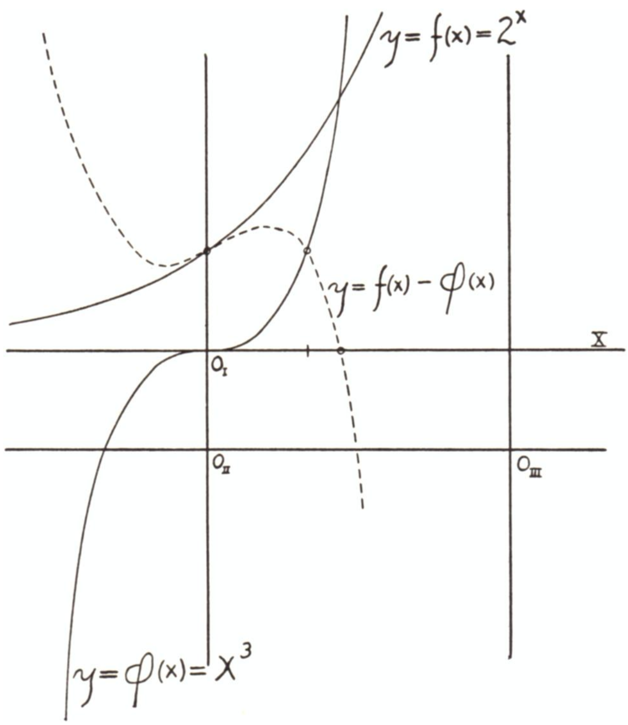

FIG. 2.

III. Given the graph of $y = f(x)$ to find the graphs of $y = f(x) \pm c$ and $y = f(x \pm c)$.

The first of these forms is clearly a special case of $y = f(x) \pm \phi(x)$ where $\phi(x)$ is a constant. The result in the

special case may however be easily determined independently of the general discussion. Obviously the ordinates of $y = f(x) \pm c$ differ from those of $y = f(x)$ by the constant value c. Hence to draw the graph of $y = f(x) \pm c$ it is sufficient simply to draw the graph of $y = f(x)$ and then to lower or raise the x-axis the distance c according as the graph desired is $y = f(x) + c$ or $y = f(x) - c$. For example, in Fig. 2 if the graph is referred to O_{II} as an origin it becomes the graph of the function $y = 2^x - x^3 + 1$. This transformation of the x-axis makes it possible in drawing the graph of a function to disregard an absolute term until the plotting is complete.

A corresponding simple transformation of the y-axis makes it possible to determine the graph of $y = f(x \pm c)$ directly from the graph of $y = f(x)$ as follows. If a is substituted for x in $f(x)$ the resulting ordinate is $f(a)$. The value of x that gives the same ordinate in $f(x + c)$ is obviously $a - c$. For the substitution of $a - c$ for x in $f(x + c)$ gives $f(a)$. Hence for every point on $f(x)$ there is a point having an equal ordinate on $f(x + c)$, this latter point being c units to the left of the corresponding point on $f(x)$. Hence to find the graph of $f(x + c)$ we plot the graph of $f(x)$ and then move the y-axis c units to the right. For example, in Fig. 2 if the graph is referred to O_{III} as origin it becomes the graph of

$$y = 2^{x+3} - (x + 3)^3 + 1 = 2^{x+3} - x^3 - 9x^2 - 27x - 26.$$

Similarly the graph of $y = f(x - c)$ may be derived from the graph of $y = f(x)$ by moving the y-axis c units to the left. As will be seen later these transformations are particularly useful in plotting the graph of the irreducible rational integral function.

IV. Given the graphs of $y = f(x)$ and $y = \phi(x)$ to find the graph of $y = f(x) \cdot \phi(x) \equiv F(x)$. (See Fig. 1.)

Since for any value of x the ordinate of $y = F(x)$ is the product of the corresponding ordinates of $y = f(x)$ and $y = \phi(x)$, when either of the latter graphs crosses the x-axis, so also does the graph of $y = F(x)$. Moreover when either of the given graphs crosses the line $y = 1$, $y = F(x)$ crosses the other. If the given graphs are on the same side of the x-axis the required graph is above the axis, otherwise it is below the axis. The position of the required graph depending on the relation-

ships between the given graphs and the lines $y=\pm 1$ are easily determined.

V. Given the graph of $y=f(x)$ to find the graphs of the following:

$$y=cf(x), \qquad y=-f(x),$$

$$y=f(cx), \qquad y=f(-x).$$

Since the ordinates of $y=cf(x)$ are c times those of $y=f(x)$ the following device may be used for deriving the graph of the former from that of the latter. When c is positive plot $y=f(x)$ using on the y-axis a unit c times as great as that used on the x-axis; then interpret the graph as though the unit on the y-axis equalled that on the x-axis and the result is the graph of $y=cf(x)$. If c is negative it is simply necessary to plot $y=|c|\cdot f(x)$ in the way outlined above but with positive ordinates plotted downward and negative ordinates upward. In particular to discover the graph of $y=-f(x)$ it is sufficient simply to plot $y=f(x)$ and then, rotating the paper 180 degrees on the x-axis, to examine the graph *through* the paper by holding it toward the light. $y=f(x)$ and $y=-f(x)$ are symmetrical to each other with respect to the x-axis.

By a process of reasoning similar to that used above it is clear that $y=f(cx)$ may be derived from $y=f(x)$ by changing the unit, or direction, or both on the x-axis before plotting. In particular the graphs of $y=f(x)$ and $y=f(-x)$ are symmetrical to each other with respect to the y-axis and either may be derived from the other by a rotation of 180 degrees about that axis.

It is clear then that to derive $y=-f(-x)$ from $y=f(x)$ it is simply necessary to turn the graph upside down.

Before proceeding farther it may be pointed out that in all but the simplest cases it will be found useful before plotting to arrange a scheme of the work. In making the scheme one naturally starts with the required form and proceeds step by step to simplify it. The actual work of plotting proceeds in the opposite direction. For example, the scheme given below was built from the bottom upward but the plotting was done in the reverse order. As in the case of the other illustrations in the paper, a simple example is used. It is clear however that the

methods may be applied to forms that are much more difficult. The graph is given in Fig. 3.

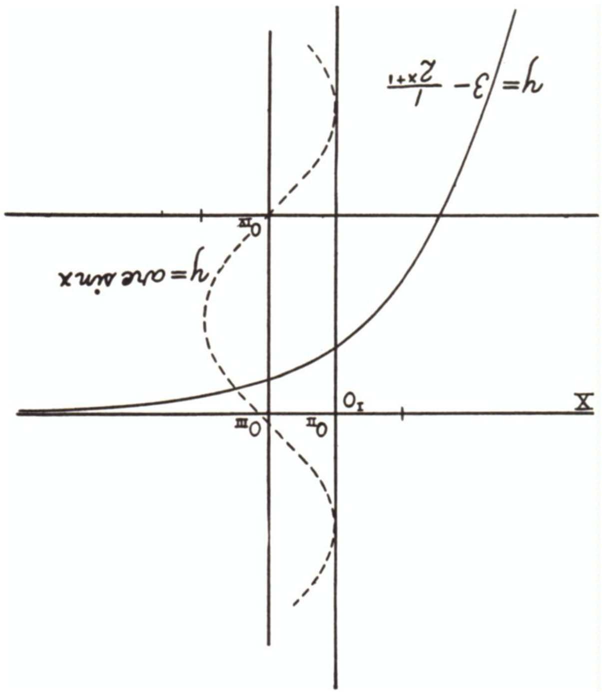

FIG. 3.

SCHEME OF WORK FOR PLOTTING $y = 3 - \frac{1}{2^{x+1}}$.

Step.	Result.
Plot $y = 2^x$	$y = 2^x$
Turn paper upside down	$y = -\frac{1}{2^x}$
Move y-axis one unit to the right	$y = -\frac{1}{2^{x+1}}$
Move x-axis three units down	$y = 3 - \frac{1}{2^{x+1}}$

VI. Given the graphs of $y = f(x)$ and $y = \phi(x)$ to find the graph of

$$y = \frac{f(x)}{\varphi(x)}.$$

Perhaps the simplest method of attacking this problem is to change the quotient into the product of the dividend and the reciprocal of the divisor. The work may then be completed as in IV above. We therefore consider the following.

VII. Given the graph of $y = f(x)$ to find the graph of

$$y = \frac{1}{f(x)}.$$

(See Fig. 4.)

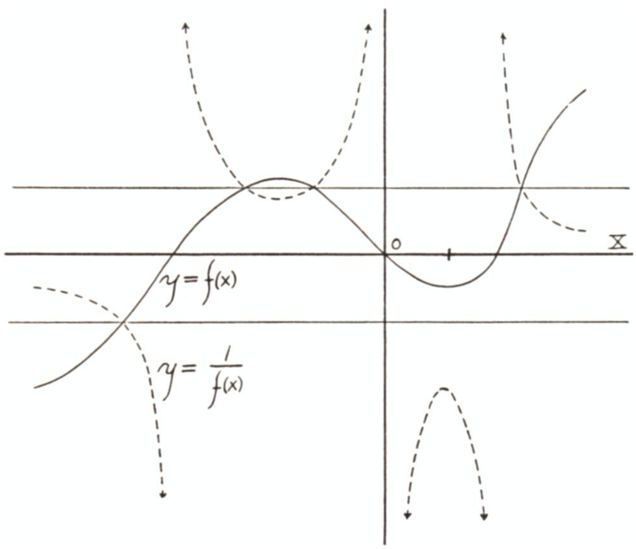

FIG. 4.

As the ordinates of the graph of the required function are the reciprocals of the corresponding ordinates of the given function, it is clear that whenever the graph of $y = f(x)$ crosses the x-axis the graph of

$$y = \frac{1}{f(x)}$$

goes to infinity and vice versa. The signs of the corresponding ordinates of the two graphs are the same. They intersect on the lines $y = \pm 1$. Points between these two lines on either graph correspond to points outside on the other.

VIII. Given the graph of $y=f(x)$ to find the graph of

$$y=f\left(\frac{1}{x}\right)$$

(See Fig. 5.)

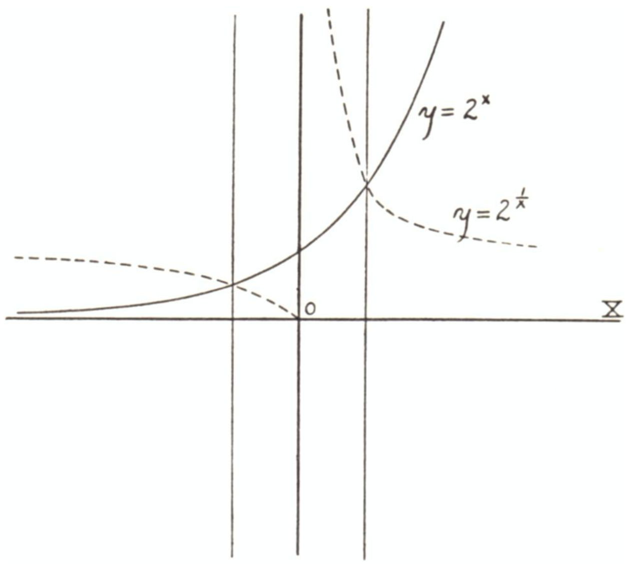

FIG. 5.

For every ordinate which the graph of $y=f(x)$ has between the lines $x=\pm 1$, the graph of

$$y=f\left(\frac{1}{x}\right)$$

has an equal corresponding ordinate outside the lines and vice versa. For if $f(x)$ becomes $f(a)$ when $x=a$, $f(1/x)$ becomes $f(a)$ when $x=1/a$. Speaking roughly, the parts of the graph of $y=f(x)$ exterior to the two lines are reversed in direction and crowded within the lines in the graph of $y=f(1/x)$. Similarly the parts of the graph of $y=f(x)$ within the lines are reversed in direction and spread over the plane outside the lines in the graph of $y=f(1/x)$.

IX. Given the graph of $y=f(x)$ to find the graph of $y=\sqrt{f(x)}\equiv F(x)$. (See Fig. 6, in which the graphs of $y=(x-1)(x-3)$ and $y=\sqrt{(x-1)(x-3)}$ are shown.)

Since the ordinates of the graph of $y=F(x)$ are the square roots of the corresponding ordinates of $y=f(x)$, the two curves meet on the x-axis and on the line $y=1$. Between the lines $y=0$ and $y=1$, the graph of $y=\sqrt{f(x)}$ lies above the graph of $y=f(x)$. Above the line $y=1$ the graph of $y=\sqrt{f(x)}$ is below the graph of $y=f(x)$. That is to say, the graph of $y=\sqrt{f(x)}$ is always between the graph of $y=f(x)$ and the line $y=1$.

If the graph of $y=f(x)$ passes below the x-axis the graph of $y=\sqrt{f(x)}$ becomes discontinuous. This leads to certain interesting results. If, for example, the term $\sqrt{-x^2}$ is added to any function the real graph of the function is thereby destroyed save for the points where it crosses the y-axis. Similarly $\sqrt{x}$ or $\sqrt{-x}$ when added to a function destroys the half of the graph to the left or right of the y-axis and transforms to a certain extent what remains. More generally the real graph of a function may be destroyed in the interval from a to b by adding the term $\sqrt{(x-a)(x-b)}$ or outside the interval by add-

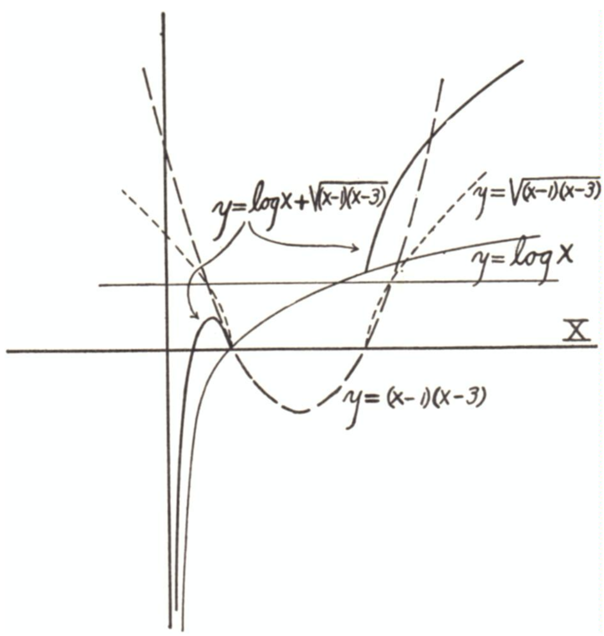

FIG. 6.

ing $\sqrt{(a-x)(x-b)}$. The graph is of course transformed to a certain extent where it is not destroyed since a term has been added. This transformation may however be reduced practically to nothing for regions ordinarily plotted by placing the radical over a large denominator, as for example, 1000^{1000}. As an illustration the graph of $y = \log x + \sqrt{(x-1)(x-3)}$ is given. (See Fig. 6.) The scheme of plotting will be clear from the figure. By introducing additional factors under the radical the graph may be destroyed throughout various intervals. It may also be destroyed in part or entire save for a set of isolated points by adding terms of the following type: $\sqrt{x} \sin nx$, $\sqrt{-x} \sin nx$, $\sqrt{-x^2} \sin nx$. As an example the graph of

$$y = \frac{x^2}{16} + \sqrt{-x^2} \sin 3x$$

is given. (See Fig. 7.) The graph consists simplv of the isolated points marked by small circles.

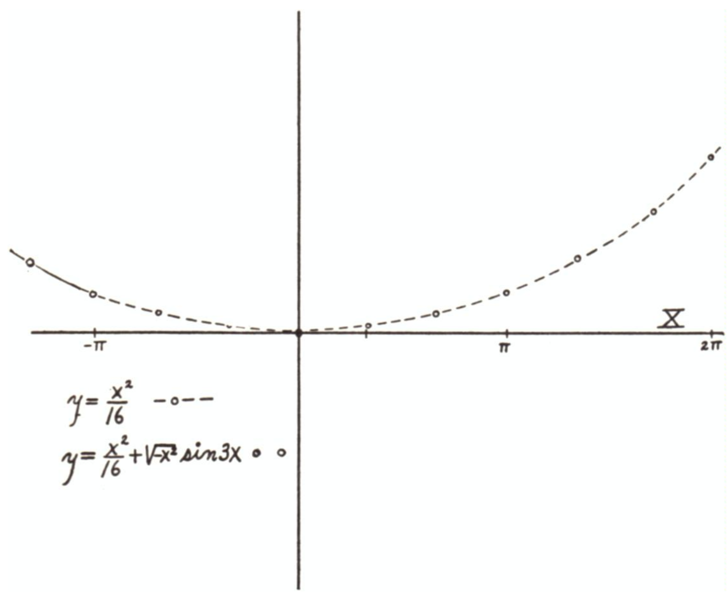

FIG. 7.

X. Given the graph of $y = f(x)$ to find the graph of $x = f(y)$. The required graph differs from the given only in that the

x and y are interchanged. If $y=f(x)$ is plotted the form of $x=f(y)$ can be discovered by rotating the paper 180 degrees about the x-axis, turning it in its own plane 90 degrees about the origin in the positive direction of rotation, and then looking through the paper toward the light. This method is useful in deriving the graphs of inverse functions.

In closing it may be noticed that irreducible rational integral functions, save those of the first, the second, and possibly the third degrees, are troublesome forms to plot by any method. The process of evaluating the function by synthetic division for a sufficient number of values of x to give a fair approximation to the curve is long and tiresome. The determination of maxima and minima is troublesome, since the degree of the first derivative is but one less than the degree of the original function. The application of certain of the methods described above may possibly shorten the amount of work in some cases. The two

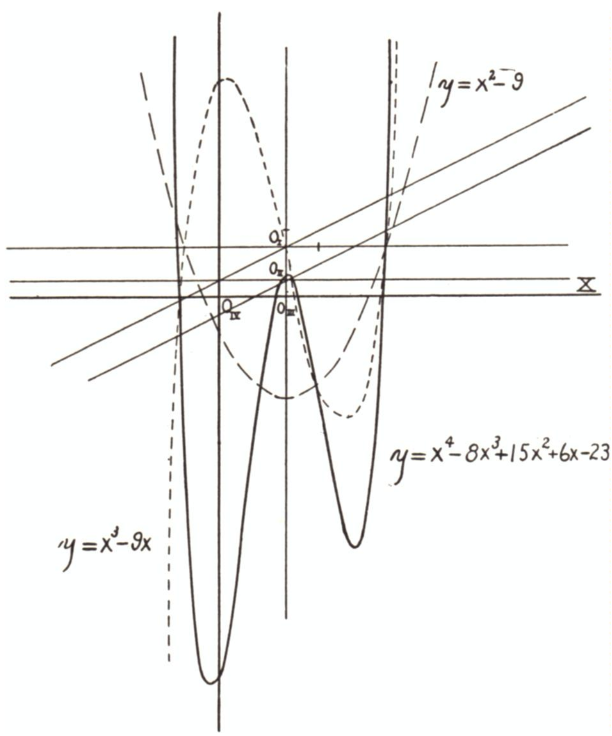

FIG. 8.

points of method that are most useful are, first, the movement of the x-axis to eliminate the absolute term and, second, the movement of the y-axis to eliminate any other term save that of highest degree. To determine the distance the y-axis must be moved in order to eliminate a certain term, $x+c$ is substituted for x, and the new function having been arranged in powers of x the coefficient of the term in question is equated to zero and the value of c determined. Fig. 8 gives the graph of

$$y=x^4-8x^3+15x^2+6x-23,$$

the unit on the y-axis being but half that on the x. The scheme used in plotting this graph is given below and also a second optional scheme suggesting another method that may in some cases be helpful in plotting the graphs of irreducible rational integral functions.

Scheme for Plotting $y=x^4-8x^3+15x^2+6x-23$.

Step.	Result.
Plot $y=x^2-9$ and $y=x$ and multiply..	$y=x(x^2-9)=x^3-9x$
Lower x-axis two units	$y=x^3-9x+2$
Plot $y=x$ and multiply graphs	$y=x^4-9x^2+2x$
Lower x-axis one unit	$y=x^4-9x^2+2x+1$
Move y-axis two units to the left	$y=x^4-8x^3+15x^2+6x-23$

Optional Scheme for Plotting $y=x^4-8x^3+15x^2+6x-23$.

Step.	Result.
Plot $y=x^4$.	
Move y-axis two units to left,	$y=f(x-2)=x^4-8x^3+24x^2-32x+16$
Plot $y=-9x^2+38x-39$.	
Add the two latter graphs..	$y=x^4-8x^3+15x^2+6x-23$

One additional illustration is given of the construction of a graph which to a student ignorant of methods of determining singular points might present considerable difficulty unless attacked by some of the methods described above. (See Fig. 9.)

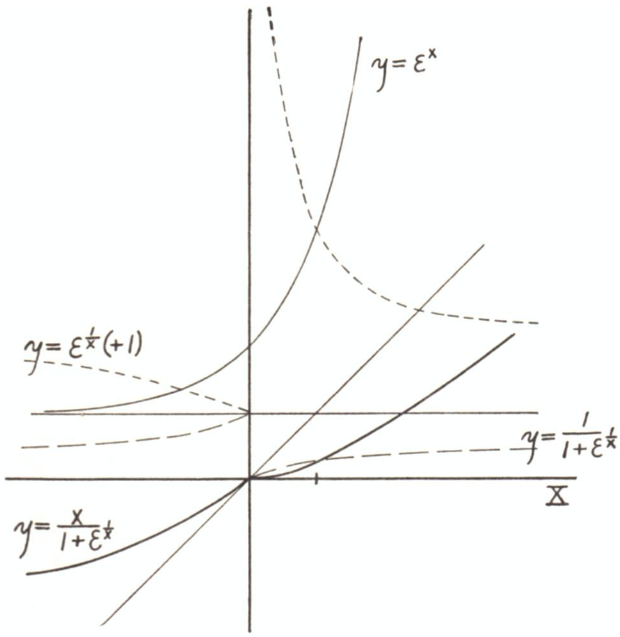

FIG. 9.

SCHEME OF PLOTTING $y=\dfrac{x}{1+e^{1/x}}$.

Step.	Result.
Plot $y=f(x)=e^{x}$.	
Plot $y=f\left(\dfrac{1}{x}\right)$	$y=e^{1/x}$
Lower x-axis one unit	$y=e^{1/x}+1$
Plot $y=\dfrac{1}{f(x)}$	$y=\dfrac{1}{1+e^{1/x}}$
Plot $y=x$ and multiply by above	$y=\dfrac{x}{1+e^{1/x}}$

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